Exercise Sheet 6

December 26, 2012

1. For all $n \in \mathbb{N}$ let f_n be an holomorphic function in a domain U, and for all $n \in \mathbb{N}$ let M_n be a positive number such that for all $z \in U$: $|f_n(z)| \leq M_n$. Also assume that $\sum_{n=1}^{\infty} M_n$ is convergent. Using Morera's Theorem, show that $\sum_{n=1}^{\infty} f_n$ is holomorphic in U.

2. Note: the complex power w^z is defined as follows: take the principal branch logarithm and define $w^z = \exp(z \cdot \log w)$. For $z \in \mathbb{C}$ such that $Re(z) > 0$, define:

$$\Gamma(z) = \int_0^{\infty} x^{z-1} \cdot e^{-x} dx$$

   Prove that Γ is holomorphic for $Re(z) > 0$.

3. Let f be a non-constant holomorphic function in a domain U. Show that the real part of f has no local maximum inside U.

4. Let f be a non-costant holomorphic function in a domain U. Assume $z_0 \in U$ is a local minimum of $|f|$. Show that $f(z_0) = 0$.

5. Find the Laurent series of $f(z) = \frac{1}{z-1} + \frac{1}{z-2}$ in the following regions:

   (a) $|z| < 1$

   (b) $1 < |z| < 2$

   (c) $2 < |z|$

6. Find the Laurent series of the following functions:

   (a) $\frac{1}{z^2-5z+6}$ around 2.

   (b) $(z^2 + 1)\exp(\frac{1}{z})$ around 0.

   (c) $\frac{\sin z}{z-2\pi}$ around 2π.

   (d) $\frac{1}{z^2+1}$ for $1 < |z - (1 + i)| < \sqrt{5}$.

   (e) $\frac{1}{1-z}$ for $|z| > 1$.

7. (a) $f(z)$ is a holomorphic function with a pole of order n at 0. Prove: the function $f(z^2)$ has a pole of order $2n$ in 0.

   (b) f and g are both holomorphic functions. Both functions have a zero of order 3 at z_0. What kind of singularity does $\frac{f}{g}$ has at z_0?

   (c) f and g are both holomorphic functions. Both functions have a pole of order 2 at z_0. What kind of singularity does $f \cdot g$ has at z_0?

8. Find and classify all the singularities of the following functions:

   (a) $\exp(\frac{1}{z-1})$

   (b) $\frac{\sin z}{z^3}$

   (c) $\frac{z^3}{\sin(z^2)}$

   (d) $\frac{\sin(iz)}{(\exp(z)-1)^2}$

   (e) $\exp(\exp(\frac{1}{z}))$